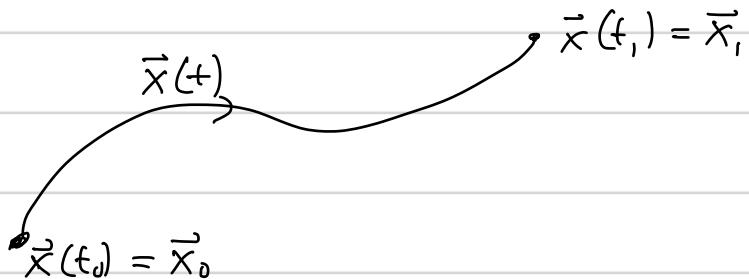


## The Propagator and the Path Integral

Let us further compare the predictions of classical and quantum mechanics on the motion of particles.

In classical mechanics, the particle moves along a trajectory



The action is a "functional" of the path :  $S[\vec{x}(t)] = \int_{t_0}^{t_1} L(\dot{\vec{x}}(t), \vec{x}(t)) dt$   
where  $L(\dot{\vec{x}}(t), \vec{x}(t)) = m \frac{\dot{\vec{x}}^2}{2} + V(\vec{x}(t))$  is the Lagrangian

The classical trajectory is the "stationary" solution  $\Rightarrow \delta S \rightarrow 0$  for small changes  $\delta \vec{x}$   
 $\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L}{\partial \vec{x}} \Rightarrow m \ddot{\vec{x}} = -\nabla V$  (Newton's Laws)

We also define canonical coords :  $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m \dot{\vec{x}}$ .

The Hamiltonian is defined :  $H(\vec{x}, \vec{p}) = \mathcal{L}(\dot{\vec{x}}, \vec{x}) - \vec{p} \cdot \dot{\vec{x}} \quad | \quad \dot{\vec{x}} = \vec{p}/m$

Hamilton's Principle function  $S[\vec{x}, \vec{p}] = \int_{t_0}^{t_1} (\vec{p} \cdot \dot{\vec{x}} - H(\vec{p}, \vec{x})) dt = \underbrace{\int_{\vec{x}_0}^{\vec{x}_1} \vec{p} d\vec{x}}_{W(\vec{x}, \vec{p})} - \int_{t_0}^{t_1} H dt$   
 $= \text{Action}[\vec{x}, \vec{p}]$

$\Rightarrow \vec{p} = \nabla S \quad \frac{\partial S}{\partial t} = -H(\vec{x}, \vec{p} = \nabla S) : \text{Hamilton Jacobi eqn.}$

We see there is explicit time dependence, energy is conserved

$$\Rightarrow S = \int_{\vec{x}_0}^{\vec{x}_1} \vec{p} \cdot d\vec{x} - E(t - t_0)$$

Note: If  $\vec{p}$  is also constant  $S = \vec{p} \cdot (\vec{x}_1 - \vec{x}_0) - E(t - t_0)$  : Look familiar ?

Quantum Mechanically, consider the transition probability to a particle to go from position  $\vec{x}_0$  @  $t=t_0$  to  $\vec{x}_1$  @  $t=t_1$ ,

$$|\psi(t_0)\rangle = |\vec{x}_0\rangle, \quad |\psi(t_1)\rangle = \hat{U}(t_1 - t_0) |\psi(t_0)\rangle$$

$$\Rightarrow \text{Prob amplitude } \langle \vec{x}_1 | \psi(t_1) \rangle = \langle \vec{x}_1 | \hat{U}(t_1 - t_0) | \vec{x}_0 \rangle = K(\vec{x}_1, \vec{x}_0; t_1 - t_0)$$

≡ Propagator

The propagator is the "position representation" of the time-evolution operator

$$\text{Generally give } |\psi(t_0)\rangle : |\psi(t)\rangle = U(t - t_0) |\psi(t_0)\rangle$$

$$\begin{aligned} \Rightarrow \psi(\vec{x}, t) &= \langle \vec{x} | \psi(t) \rangle = \langle \vec{x} | U(t - t_0) |\psi(t_0)\rangle = \int d^3x' \langle \vec{x} | \hat{U}(t - t_0) | \vec{x}' \rangle \langle \vec{x}' | \psi(t_0) \rangle \\ &= \int d^3x' K(\vec{x}, \vec{x}'; t - t_0) \psi(\vec{x}', t_0) \end{aligned}$$

To begin, consider propagation an infinitesimal  $\Delta t$  in 1D

$$\begin{aligned} K(x_0, x_0 + \Delta x, \Delta t) &= \langle x_0 + \Delta x | e^{-\frac{i}{\hbar} \Delta t (\frac{p^2}{2m} + V(x))} | x_0 \rangle \underset{\text{to lowest order in } \Delta t}{\approx} \langle x_0 + \Delta x | e^{-\frac{i \Delta t}{\hbar} \frac{p^2}{2m}} e^{-i \frac{\Delta t}{\hbar} V(x)} | x_0 \rangle \\ &= \int_{-\infty}^{\infty} dp \langle x_0 + \Delta x | p \rangle \langle p | x_0 \rangle e^{-\frac{i \Delta t}{\hbar} \frac{p^2}{2m}} e^{-i \frac{\Delta t}{\hbar} V(x)} = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{i \frac{\Delta x p}{\hbar}} e^{-i \frac{\Delta t}{\hbar} \frac{p^2}{2m}} e^{-i \frac{\Delta t}{\hbar} V(x)} \end{aligned}$$

$$\text{And } \int_{-\infty}^{\infty} dp e^{-\alpha(p - \beta)^2} = \sqrt{\frac{\pi}{2}} \quad \text{Re } \alpha \geq 0 \quad \text{or} \quad (\text{Re } \alpha = 0 \text{ and } \text{Im } \alpha \neq 0)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{i \frac{\Delta x p}{\hbar}} e^{-i \frac{\Delta t}{\hbar} \frac{p^2}{2m}} = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{-\frac{i \Delta t}{2m} (p - \frac{\Delta x}{\Delta t} m)^2} e^{i \frac{m \Delta x^2}{2\Delta t}} = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} e^{-\frac{i m \Delta x^2}{2\Delta t}}$$

$$\Rightarrow K(x_0 + \Delta x, x_0; \Delta t) = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} e^{-\frac{i \Delta t}{\hbar} \left( \frac{m \Delta x^2}{2\Delta t} - V(x) \right)} = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} e^{\frac{i}{\hbar} \Delta t L(x, \dot{x})}$$

where  $L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$  is the Lagrangian

$$\Rightarrow K(x_0 + \Delta x, x_0, \Delta t) = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} e^{\frac{i}{\hbar} \underset{\text{Action}}{\int} S(x_0, x_0 + \Delta x; t)}$$

This is a beautiful result, first noted by Dirac.

## The propagator as a transition amplitude

There is a particularly useful way to write the propagator:

$$\text{Let } |x(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} |x\rangle \text{ (time-evolved position basis)}$$

$$\text{Note } \int dx(t) |x(t)\rangle \langle x(t)| = \mathbb{1} \quad \forall t$$

We can thus write the propagator  $K(x_N, x_0; t_N - t_0) = \langle x_N(t_N) | x_0(t_0) \rangle$

Note, we can always insert a complete set at some intermediate time  $t_N < t_i < t_0$

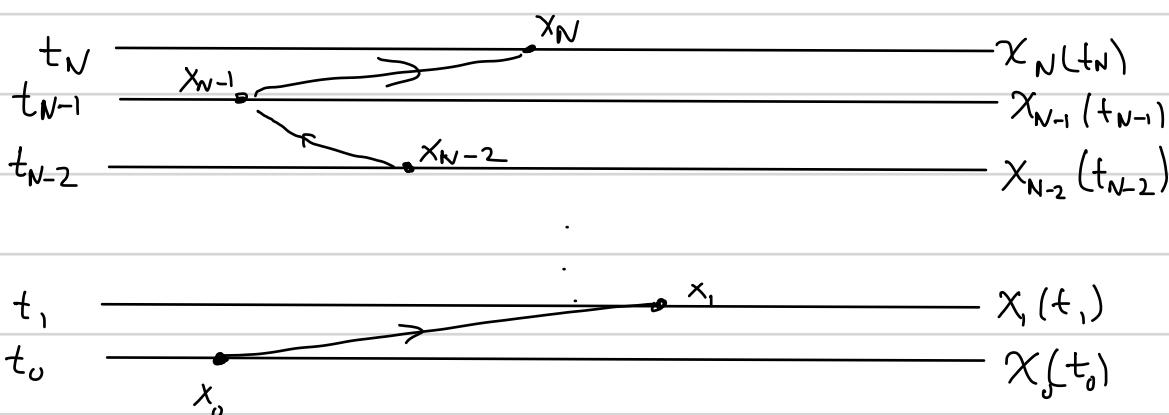
$$\langle x_N(t_N) | x_0(t_0) \rangle = \int_{-\infty}^{\infty} dx_i \langle x_N(t_N) | x_i(t_i) \rangle \langle x_i(t_i) | x_0(t_0) \rangle$$

$$\Rightarrow K(x_N, x_0; t_N - t_0) = \int_{-\infty}^{\infty} dx_i K(x_N, x_i; t_N - t_i) K(x_i, x_0; t_i - t_0)$$

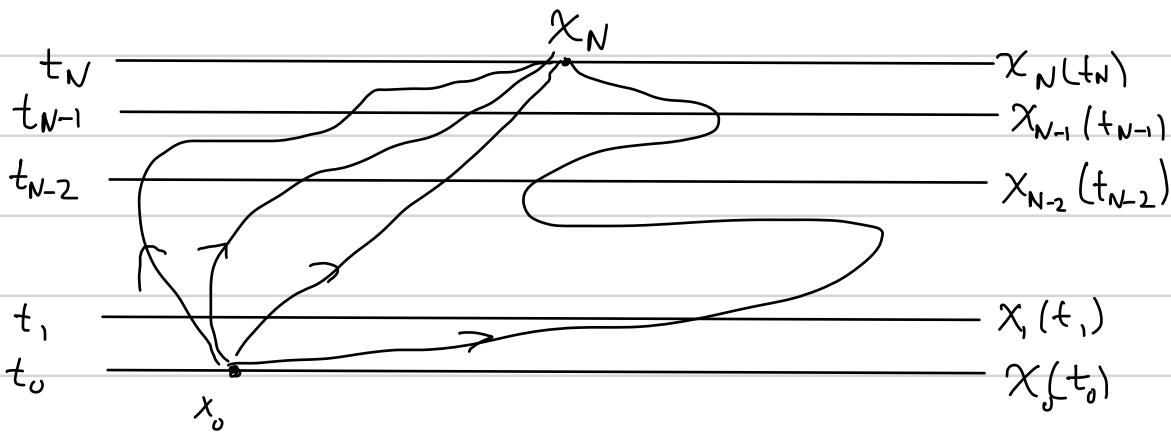
Consider then, chopping up the time interval  $t_N - t_0$  into  $N$ -slices  $\Delta t = \frac{t_N - t_0}{N}$ . Ultimately we want to take the limit  $N \rightarrow \infty$ ,  $\Delta t \rightarrow 0$

$$\begin{aligned} \Rightarrow K(x_N, x_0; t_N - t_0) &= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dx_{N-2} \cdots \int_{-\infty}^{\infty} dx_1 \langle x_N(t_N) | x_{N-1}(t_{N-1}) \rangle \langle x_{N-1}(t_{N-1}) | x_{N-2} \rangle \cdots \langle x_1(t_1) | x_0(t_0) \rangle \\ &= \lim_{N \rightarrow \infty} \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} dx_i \underbrace{\prod_{i=0}^{N-1} K(x_{i+1}, x_i; \Delta t)}_{\sqrt{\frac{m}{2\pi\hbar i \Delta t}}} = \lim_{N \rightarrow \infty} \prod_{i=1}^{\infty} \int_{-\infty}^{\infty} dx_i \sqrt{\frac{m}{2\pi\hbar i \Delta t}} e^{\frac{i}{\hbar} \sum L(x_i, \dot{x}_i) \Delta t} \end{aligned}$$

The integral over all of the intermediate positions can be understood in a space-time diagram:



In the limit  $N \rightarrow \infty$  this sum over intermediate positions becomes an integral over all possible continuous paths



In this limit,  $\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} L(x_i, \dot{x}_i) = \int_{t_0}^{t_N} dt L(x(t), \dot{x}(t)) = S[x(t); t_N - t_0]$   
 action for some path

$$\Rightarrow K(x_N; x_0; t_N - t_0) = N \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar} S[x(t); t_N - t_0]}$$

↑ measure over "space of paths"

$N$  = normalization constant depending on measure

This is the path-integral form of the propagator first derived by Feynmann in 1948 in his PhD thesis. It gives a beautiful picture with which to understand quantum dynamics. Classically the trajectory that takes us from  $x_0 @ t_0$  to  $x_N @ t_N$  is the one such that the action is stationary to small changes  $S[x_{\text{classical}}(t), t_N - t_0] = 0$ . Quantum mechanically, all paths contribute to the probability to a particle localized @  $x_0$  will be found @  $x_N$  at a later time. Each path contributes a probability amplitude depending on the action  $e^{i S[x(t)]/\hbar}$ . We see the classical limit arising when  $S[x(t)] \gg \hbar$  for all paths. Then only the path with  $S = 0$  will contribute as all others will rapidly oscillate and tend to destructively interfere. The path integral is beautiful conceptually, but is rarely calculatively useful. It is, however a foundational method in quantum field theory, as space and time are treated on equal footing. In addition, it is useful in statistical physics in calculations of the partition function.